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JBD-003-1161001

Seat No. ______

M. Sc. (Sem. I) (CBCS) Examination

December - 2019

Mathematics : CMT - 1001

(Algebra - I)

Faculty Code : 003
Subject Code : 1161001

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (1) All questions are compulsory.
(2) Each question carries 14 marks.

1 Answer any **seven** questions : **7×2=14**

(i) Write down two subgroups of S_3 which are not normal, where $S_3 = \{e, \sigma, \sigma^2, \psi, \sigma\psi, \sigma^2\psi\}$.

(ii) Define a simple group and give an example of a simple group. Is A_4 a simple group ? (Y/N).

(iii) Prove or disprove that S_3 is a simple group.

(iv) Define an ideal I of a ring R. Let $a, b, c \in I$. Deduce that $a - b - 2c \in I$.

(v) Let G be a finite group and a prime p divide to o(G). Define a p-Sylow subgroup of G.

(vi) Let A, B, C ideals of a ring R. Prove that $A \cap B \cap C$ is also an ideal of R.

(vii) Let G be a finite group with o(G) = 147. Write down order of 3-Sylow and 7-Sylow subgroups of G.

(viii) Define a prime ideal of a ring R. Is all prime ideals of $(\mathbb{Z}, +, \cdot)$ are maximal ideals ? Justify.

2 Answer any **two** questions : **2×7=14**

(a) State and prove Third Fundamental Theorem of Groups.

(b) Let G be a group and

$$G' = \left\{\prod_{i-1}^{t} a_i b_i a_i^{-1} b_i^{-1} / a_i, b_i \in G, \forall i = 1, 2, \ldots, t\right\}$$ be the

commutator subgroup G. In standard notation prove that G' is a normal subgroup of G and G/G' is an abelian group.

(c) Let G be a non-abelian group of order 6. Prove that G is isomorphic to S_3 .

3 Answer any **one** question : **1×14=14**

(a) (i) State and Prove Sylow's Third Theorem.

(ii) Let G be a finite abelian group and a prime p divide to $o(G)$. Let P be a Sylow p-subgroup of G. Prove that P is only Sylow p-subgroup of $G \Leftrightarrow P$ is normal subgroup of G.

(b) Let R be a ring. Prove that for any positive integer n, any ideal of $M_n(R)$, the ring of all the nxn matrices over R is given by $M_n(I)$, where I ranges through all the ideals of R.

(c) Prove that $A_n\left(n \geq 5\right)$ is a simple group. For $n \geq 5$, prove that the collection of all normal subgroups of S_n is $\left\{\{e\}, A_n, S_n\right\}$.

4 Answer any **two** questions : **2×7=14**

(a) State and Prove First Isomorphism Theorem of Rings.

(b) Let A, B be two ideals of a ring R. Define product AB and sum $A + B$ of two ideals in R. Prove that AB, $A + B$ and $AB \cap \left(A+B\right)$ all are ideals of R.

(c) Let $f : R \to T$ be an onto ring homomorphism. Let $\mathcal{C}$ the collection of ideals of R which contains ker f and $\mathcal{D}$ be the collection of all ideals of T. Prove that there is a bijective map from $\mathcal{C}$ into $\mathcal{D}$.

5 Answer any **two** questions : **2×7=14**

(a) Let G be a finite group, with $O\left(G\right) = p \cdot q$, where p and q both are primes ($p < q$). If $p + q - 1$, then prove that G must be a cyclic group.

(b) Let R be a commutative ring and M be an ideal of R. Prove that M is a maximal ideal of R if and only if R/M is a field.

(c) Let G be a group and N_i be normal subgroups of G, $\forall\, i = 1, 2, \ldots.., n$. Prove that G is the internal direct product of $N_1, N_2, \ldots., N_n$ iff $G = N_1\, N_2 \,..\, N_n$ and $N_i \cap N_1 \,..\, N_{i-1}\, N_{i+1} \,..\, N_n = \left\{e\right\}$, for every $i \in \left\{1, 2, \ldots., n\right\}$.

(d) Prove that :

(i) Every irreducible element of a Principle Ideal Domain R is always a prime element of R and

(ii) Every Euclidean Domain is also Principle Ideal Domain.